

Acyclic systems of permutations and fine mixed subdivisions of simplices

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Abstract

A fine mixed subdivision of a $(d-1)$ -simplex T of size n gives rise to a system of $\binom{d}{2}$ permutations of $[n]$ on the edges of T , and to a collection of n unit $(d-1)$ -simplices inside T . Which systems of permutations and which collections of simplices arise in this way? The *Spread Out Simplices Conjecture* of Ardila and Billey proposes an answer to the second question. We propose and give evidence for an answer to the first question, the *Acyclic System Conjecture*.

We prove that the system of permutations of T determines the collection of simplices of T . This establishes the Acyclic System Conjecture as a first step towards proving the Spread Out Simplices Conjecture. We use this approach to prove both conjectures for $n = 3$ in arbitrary dimension.

1 Introduction

The fine mixed subdivisions of a dilated simplex arise in numerous contexts, and possess a remarkable combinatorial structure, which has been the subject of great attention recently. The goal of this paper is to prove several structural results about these subdivisions.

A fine mixed subdivision of a $(d-1)$ -simplex T of size n gives rise to a system of $\binom{d}{2}$ permutations of $[n]$ on the edges of T , and to a collection of n unit $(d-1)$ -simplices inside T . We address the question: **Which systems of permutations and which collections of simplices arise from such subdivisions?** We prove several results in this direction. In particular we prove Ardila and Billey's Spread Out Simplices Conjecture [2, Conjecture 7.1] in the special case $n = 3$.

1. Introduction. We begin by summarizing the different sections of the paper, and stating our main results and conjectures. Figure 1 illustrates the main concepts with pictures of the case $d = 3$. We delay the precise definitions until the later sections.

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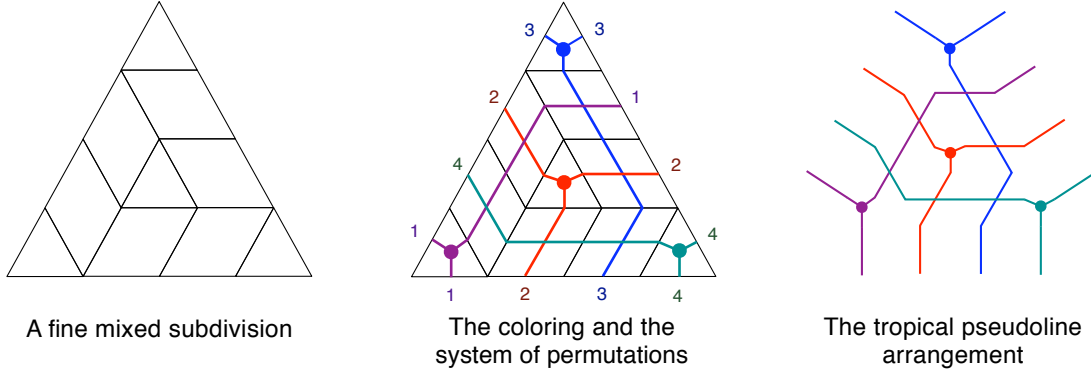


Figure 1: A fine mixed subdivision of $4\Delta_2$.

2. The fine mixed subdivisions of a simplex $n\Delta_{d-1}$ are the subdivisions of the dilated simplex $n\Delta_{d-1}$ into fine mixed cells. A *fine mixed cell* is a $(d-1)$ -dimensional product of faces of Δ_{d-1} lying in independent affine subspaces. For $d=3$, fine mixed subdivisions are the *lozenge* tilings of an equilateral triangle into unit equilateral triangles and rhombi.

The fine mixed subdivisions of $n\Delta_{d-1}$ are in one-to-one correspondence with triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$ via the Cayley trick [18]. These and other equivalent objects arise very naturally in many contexts [1, 3, 4, 5, 8, 9, 10, 14, 16, 17, 18, 19]. Fine mixed subdivisions are our main object of study. **We will often call them simply “subdivisions”.**

3. The coloring of a fine mixed subdivision is a natural coloring of the cells of a fine mixed subdivision. It gives rise to an arrangement of tropical pseudohyperplanes which plays a key role in the theory of tropical oriented matroids. [3, 11].

4. The system of permutations of a fine mixed subdivision T is the restriction of the coloring to the edges of the simplex $n\Delta_{d-1}$. It can be seen as a set of permutations of $[n]$, one on each edge. It has the great advantage that it is simpler than the coloring, while maintaining substantial geometric information about the subdivision. Say a system of permutations of T is *acyclic* if no closed walk on the edges of the simplex contains two colors in alternating order: $\dots i \dots j \dots i \dots j \dots i \dots j \dots$. In two dimensions, this property characterizes the systems of permutations coming from subdivisions:

Theorem 4.2. (2-D Acyclic System Theorem) *A system of permutations on the edges of a triangle can be achieved by a lozenge tiling if and only if it is acyclic.*

We also show a result which will be relevant later:

Theorem 4.3. (Short version) *The positions of the triangles in a lozenge tiling of a triangle are completely determined by the system of permutations.*

5. The Acyclic System Conjecture seeks to generalize Theorem 4.2 to higher dimensions:

Theorem 5.6. *The system of permutations of a fine mixed subdivision of $n\Delta_{d-1}$ is acyclic.*

Acyclic System Conjecture 5.7. *Any acyclic system of permutations on $n\Delta_{d-1}$ is achievable as the system of permutations of a fine mixed subdivision.*

6. Duality, deletion, and contraction are useful notions, inspired by matroid theory, that were first studied for triangulations of products of simplices by Santos in [18], and for tropical oriented matroids by Ardila and Develin in [3]. We explore these notions in the context of fine mixed subdivisions and systems of permutations, showing that they are compatible with the earlier ones.

7. From systems of permutations to simplex positions. Ardila and Billey [2] proved that any fine mixed subdivision on $n\Delta_{d-1}$ contains exactly n simplices. We use duality to generalize Theorem 4.3 to any dimension:

Theorem 7.1. (Short version) *The positions of the n simplices in a fine mixed subdivision of $n\Delta_{d-1}$ are completely determined by its system of permutations.*

8. The Spread Out Simplices Conjecture of Ardila and Billey, which is motivated by the Schubert calculus computations of Billey and Vakil [6], concerns a surprising relation between fine mixed subdivisions and the matroid of lines in a generic complete flag arrangement. Using the machinery built up in the previous sections, we are able to prove this conjecture for small simplices in any dimension.

Ardila and Billey [2] showed that the n simplices in any fine mixed subdivision of $n\Delta_{d-1}$ must be *spread out*, meaning that any sub-simplex of size k contains at most k of them. They also conjectured that the converse holds:

Spread Out Simplices Conjecture 8.1. [2] *A collection of n simplices in $n\Delta_{d-1}$ can be extended to a fine mixed subdivision if and only if it is spread out.*

Theorem 7.1 allows us to split the Spread Out Simplices Conjecture 8.1 into two: the Acyclic System Conjecture 5.7 and the Weak Spread Out Simplices Conjecture 8.8:

Weak Spread Out Simplices Conjecture 8.8. *Every spread out collection of n simplices in $n\Delta_{d-1}$ can be achieved as the set of simplices of an acyclic system of permutations.*

Using this approach, we are able to show:

Theorem 8.11. *The Spread Out Simplices Conjecture holds for $n = 3$.*

2 Fine mixed subdivisions of a simplex

Remark 2.1. We will simply refer to fine mixed subdivisions as “subdivisions” throughout the paper. The only other subdivisions we will consider are the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, which we will always refer to as “triangulations.”

Remark 2.2. Throughout the paper, the vertices of Δ_{n-1} will be denoted v_1, \dots, v_n and the vertices of Δ_{d-1} will be denoted w_1, \dots, w_d . The letters a and b will represent elements of $[d]$ and the letters i, j, k , and ℓ will represent elements of $[n]$.

Fine mixed subdivisions and several equivalent objects have been recently studied from many different points of view. Aside from their beautiful intrinsic structure [4, 5, 9], they have been used as a building block for constructing efficient triangulations of high dimensional cubes [10, 14] and disconnected flip-graphs [16, 17]. They also arise very naturally in connection with root lattices [1], arrangements of flags [2], tropical geometry [3, 8, 13], transportation problems, and Segre embeddings [19].

Before defining and studying subdivisions of $n\Delta_{d-1}$ in full generality, let us start by discussing the easier – but by no means trivial – problem of understanding the lozenge tilings of an equilateral triangle. This is the special case $d = 3$.

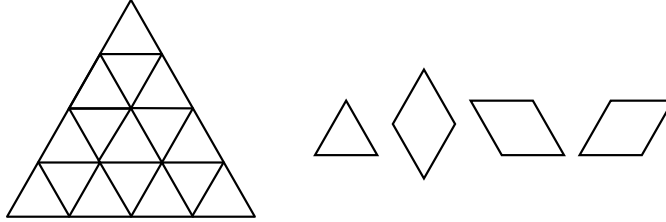


Figure 2: The triangle $4\Delta_2$ and the four different tiles allowed in a lozenge tiling.

Let $n\Delta_2$ be an equilateral triangle with side length equal to n . A *lozenge tiling* of $n\Delta_2$ is a subdivision of $n\Delta_2$ into upward unit triangles and unit rhombi, as illustrated on the right hand side of Figure 2. It is not hard to see that any lozenge tiling of $n\Delta_2$ consists of n triangles and $\binom{n}{2}$ rhombi. Figure 3 shows an example of a lozenge tiling of $4\Delta_2$.

The most natural high-dimensional analogues of the lozenge tilings of the triangle $n\Delta_2$ are the fine mixed subdivisions of the simplex $n\Delta_{d-1}$. We briefly recall their definition; for a more thorough treatment, see [18].

The *Minkowski sum* of polytopes P_1, \dots, P_k in \mathbb{R}^m is the polytope:

$$P_1 + \dots + P_k := \{p_1 + \dots + p_k \mid p_1 \in P_1, \dots, p_k \in P_k\}.$$

Let

$$\Delta_{d-1} = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ and } x_1 + \dots + x_d = 1\}$$

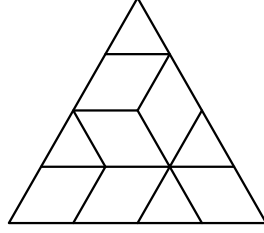


Figure 3: Example of a lozenge tiling of $4\Delta_2$.

be the standard unit $(d-1)$ -simplex, and $n\Delta_{d-1} = \Delta_{d-1} + \cdots + \Delta_{d-1}$ be its scaling by a factor of n .

A *fine mixed cell* is a Minkowski sum $B_1 + \cdots + B_n$ where the B_i s are faces of Δ_{d-1} which lie in independent affine subspaces, and whose dimensions add up to $d-1$. A *fine mixed subdivision* S of $n\Delta_{d-1}$ is a subdivision¹ of $n\Delta_{d-1}$ into fine mixed cells. Figure 4 shows examples of subdivisions of $3\Delta_2$ and $3\Delta_3$.

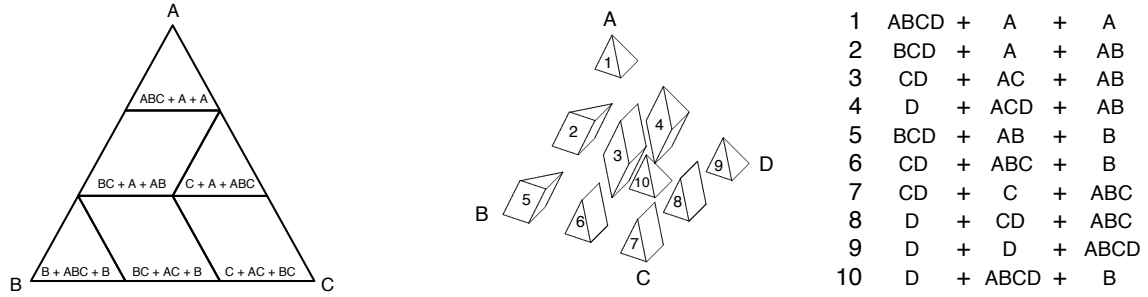


Figure 4: Subdivisions of $3\Delta_2$ and $3\Delta_3$, and the Minkowski sum decompositions of the full dimensional cells.

Remark 2.3. Santos [18] showed that the cells in a subdivision of $n\Delta_{d-1}$ can be labeled by ordered Minkowski sums in such a way that, if $B_1 + \cdots + B_n$ is a face of $C_1 + \cdots + C_n$, then B_i is a face of C_i for each i .²

Remark 2.4. Ardila and Billey showed that any subdivision of $n\Delta_{d-1}$ contains exactly n tiles that are simplices [2, Proposition 8.2].

¹A subdivision of a polytope P is a tiling of P with polyhedral cells whose vertices are vertices of P , such that the intersection of any two cells is a face of both of them.

²This property is normally required in the definition of a mixed subdivision of $P_1 + \cdots + P_n$, but it holds automatically in this special case.

3 The coloring of a fine mixed subdivision

Given a fine mixed subdivision of a simplex one can construct its *colored dual polyhedral complex*, which we simply call its *colored dual*. This complex can be regarded as a *tropical pseudo-hyperplane arrangement*; the interested reader is referred to [3, 13].

Very loosely speaking, the colored dual assigns a different color to each of the n unit simplices, and lets each color spread from the center of the simplex through the cells of the tiling. Figures 5 and 6 illustrate this process in dimensions 2 and 3.

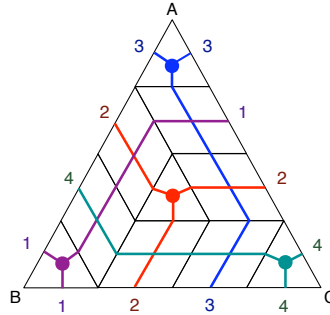


Figure 5: The colored dual of a subdivision of $4\Delta_2$ and the corresponding system of permutations on the edges of the triangle (which we will introduce later).

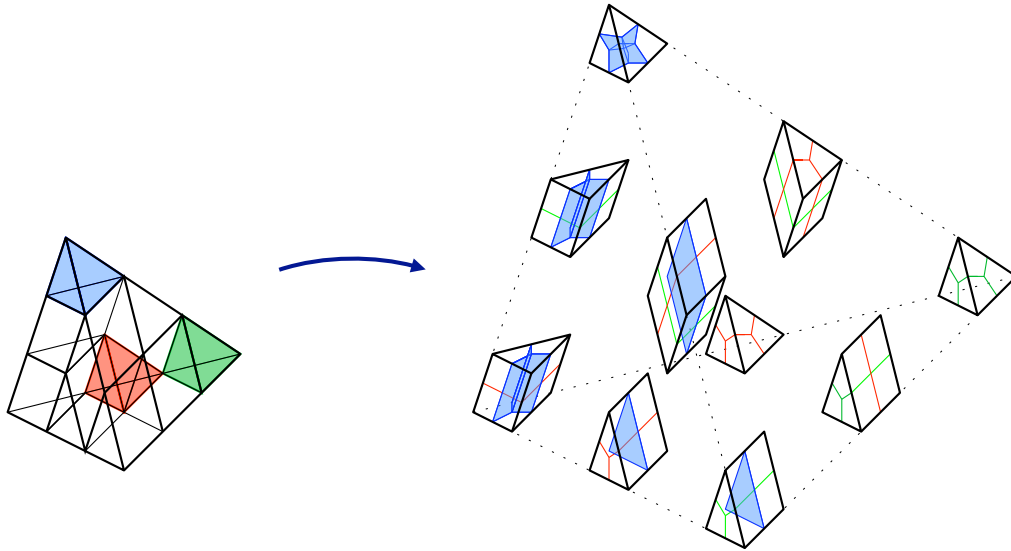


Figure 6: The colored dual of a subdivision of $3\Delta_3$. Only one color is shown in its entirety.

Formally, we define the coloring using the *mixed Voronoi subdivision*.

Definition 3.1. The *Voronoi subdivision* of a k -simplex divides it into $k + 1$ regions, where region i consists of the points in the simplex for which i is the closest vertex. Given a subdivision S of $n\Delta_{d-1}$, we subdivide each fine mixed cell $S_1 + \cdots + S_n$ into regions $R_1 + \cdots + R_n$, where R_i is a region in the Voronoi subdivision of S_i . The resulting subdivision of $n\Delta_{d-1}$ is called the *mixed Voronoi subdivision* of S .

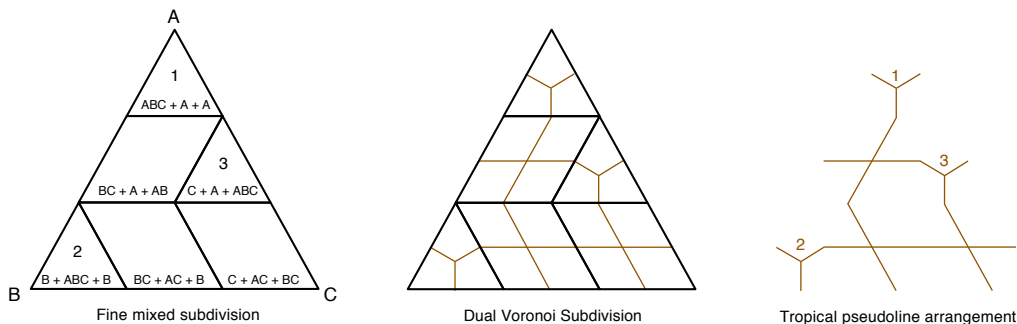


Figure 7: A fine mixed subdivision, its mixed Voronoi subdivision, and its colored dual.

Definition 3.2. The *colored dual* of a subdivision S of $n\Delta_{d-1}$ is the colored polyhedral complex formed by the lower-dimensional faces in the mixed Voronoi subdivision of S , excluding those on the boundary of $n\Delta_{d-1}$. In the mixed cell $S_1 + \cdots + S_n$, *color* i is given to $S_1 + \cdots + \overline{S_i} + \cdots + S_n$, where $\overline{S_i}$ is the complex of lower dimensional faces in the Voronoi subdivision of S_i , excluding those on the boundary of S_i .

We will not use the metric properties of this subdivision. In fact, the dual complex may be defined as a purely combinatorial object. The distinction is not important for us, but this choice of geometric realization will simplify some of our definitions.

Remark 3.3. The *color* i is the subcomplex (called a *tropical pseudohyperplane* in [3]) consisting of the cells of the colored dual having color i . It subdivides the simplex $n\Delta_{d-1}$ into d regions which are naturally labeled by the vertices w_1, \dots, w_d of the simplex.

A cell $S_1 + \cdots + S_n$ is intersected precisely by the colors i such that $\dim(S_i) \neq 0$, or equivalently, S_i has at least two letters. The summand S_i is given by the set of regions (letters) of color i that this cell intersects. (See Figure 7 for an example). It will be useful to think that a face of the triangulation has the same color(s) as its dual cell in the colored dual.

Remark 3.4. In the colored dual of a lozenge tiling S , every pair of colors intersects exactly once. One way to see this is to consider, for positive variables $\lambda_1, \dots, \lambda_n$, the subdivision S_λ of the triangle $\lambda_1\Delta_2 + \cdots + \lambda_n\Delta_2$ which is combinatorially isomorphic to S . The area of this triangle is $\sum_i \frac{1}{2}\lambda_i^2 + \sum_{i < j} \lambda_i\lambda_j$. The i th triangle of S_λ contributes an

area of $\frac{1}{2}\lambda_i^2$ to the subdivision, while a rhombus where colors i and j intersect contributes an area of $\lambda_i\lambda_j$. Therefore there is exactly one such rhombus for each i and j . A similar statement (and proof) holds in any dimension.

4 The system of permutations: the two-dimensional case

In this section, which focuses only on the two-dimensional case, we define our main object of study: the system of permutations of a subdivision. We then prove several structural results about these systems of permutations. In Sections 5, 6, and 7 we will do this (somewhat less successfully) in higher dimensions.

Given a lozenge tiling of the triangle $n\Delta_2$, and a numbering of its n triangles, restrict the colored dual to the edges. This determines three permutations of $[n]$, which we read in clockwise direction, starting from the lower left vertex. In Figure 5, the system of permutations is (1423, 3124, 4321).

We address three questions:

1. Is a lozenge tiling completely determined by its system of permutations?
2. Which triples of permutations of $[n]$ can arise from a lozenge tiling in this way?
3. How are the system of permutations and the positions of the unit triangles related?

The answer to Question 1 is negative, as Figure 8 shows. Questions 2 and 3 are more interesting, and they are addressed in the following three subsections. We answer Question 2 positively: In Theorem 4.2 we give a simple characterization of the systems of permutations that can be obtained from a lozenge tiling. We also answer Question 3 by showing, in Theorem 4.3, that the system of permutations determines uniquely the numbered positions of the triangles.

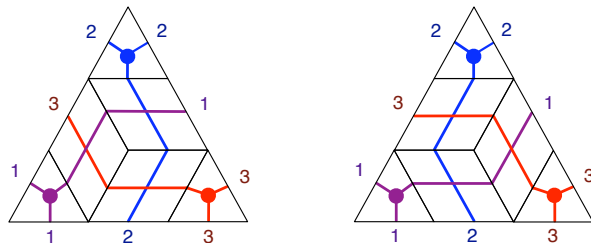


Figure 8: Two different tilings with the same system of permutations.

4.1 The two-dimensional acyclic system theorem.

Definition 4.1. A *system of permutations* on the edges of the triangle $n\Delta_2$ is a set of three permutations of $[n]$ on the edges of the triangle. We say that a system of permutations is *acyclic* if, when we read the three permutations in clockwise direction, starting from a vertex of the triangle, we never see a “cycle” of the form $\dots i \dots j \dots, \dots i \dots j \dots, \dots i \dots j \dots$.

The system of permutations $(12, 12, 12)$ on the edges of $2\Delta_2$ is the smallest system that is not acyclic. It clearly cannot be realized as the system of permutations of a lozenge tiling.

Theorem 4.2 (2-D Acyclic System Theorem). *Let σ be a system of permutations on the edges of the triangle $n\Delta_2$. Then σ is achievable as the system of permutations of a lozenge tiling if and only if σ is acyclic.*

Proof of Theorem 4.2. Let $\sigma = (u, v, w)$ be a system of permutations of a lozenge tiling of the triangle $n\Delta_2$. If the three permutations u, v , and w contained the elements i and j in the same order, then in the dual complex, colors i and j would need to intersect at least twice, contradicting Remark 3.4. This proves the forward direction.

For the converse we proceed by induction. The case $n = 1$ is trivial. Now assume that the result is true for $n - 1$, and consider an acyclic system $\sigma = (u, v, w)$ of permutations of $[n]$. Let $\sigma' = (u', v', w')$ be the acyclic system of permutation of $[n - 1]$ obtained by removing the number n from u, v , and w , and let T' be a tiling of the triangle ABC (of side length $n - 1$) realizing σ' . Let D, E , and F be the points on the segments BC, CA , and AB where the number n needs to be inserted in the permutations w', v' , and u' .

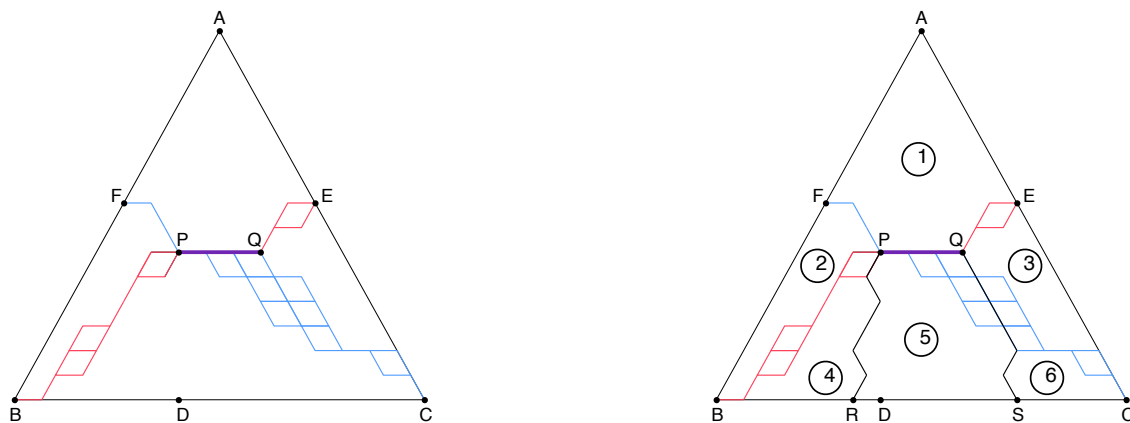


Figure 9: Left: The southwest paths from E to B and the southeast paths from F to C . Right: The paths PR, QS, G_B , and G_C divide the triangle into six regions.

In the tiling T' , let G_B be the union of all “southwest” paths from E to B , consisting of southwest and west edges. Let G_C be the union of all “southeast” paths from F to C ,

consisting of southeast and east edges. The (non-empty) intersection of G_B and G_C is a horizontal segment (or possibly a single point); label its left and right endpoints P and Q respectively. Assume that T' was chosen so that the length of PQ is maximum.

Now consider the leftmost “south” path, using southwest and southeast edges, from P to edge BC . Let its other endpoint be R , and call this path PR . Similarly, let QS be the rightmost “south” path from Q to edge BC . The previous paths split the triangle into six regions, which we number $1, \dots, 6$ as shown in the right panel of Figure 9. The cells in G_C and G_B (which are necessarily rhombi) are considered to be in none of the six regions.

If D is between R and S , then there is a “north” path, using northeast and northwest edges, from D to a point M on PQ . Consider any northwest path MF along G_B and any northeast path ME along G_C . Now cut the tiling along the paths MD , ME , and MF , and glue it back together using an equilateral triangle at M and three paths of rhombi of the shape of MD , ME , and MF , as shown in Figure 10. The result will be a tiling T which realizes the system of permutations σ .

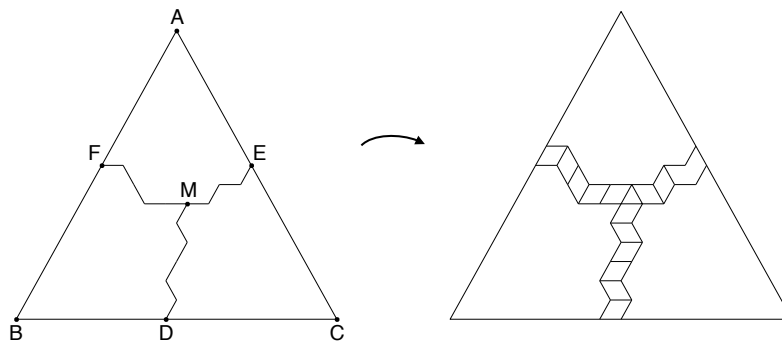


Figure 10: From a tiling of $(n-1)\Delta_2$ to a tiling of $n\Delta_2$.

If D is not between R and S , then we claim that σ is not acyclic. To prove it, assume without loss of generality that D is to the left of R . Consider the edge of T' directly to the left of R ; say it has color i . It is clear that triangle i must be in region 1, 2, or 4. We will show that in fact triangle i is in region 2. This will imply that σ contains the cycle $\dots i \dots n \dots, \dots i \dots n \dots, \dots i \dots n \dots$.

First we show that triangle i is not in region 4. Let PB be the southernmost path from P to B in G_B , and let P' be the first place where the paths PB and PR diverge. Note that all edges from P to P' must be southwest edges. By the definition of PR , there are no southwest edges hanging from P' . In particular, the next edge in PB after P' is a west edge. This forces all the tiles directly to the left of PR to be horizontal. Therefore triangle i is above P' , and hence not in region 4.

Now assume that triangle i is in region 1. Then color i must enter and exit region 2 by crossing horizontal edges. This forces all tiles in region 2 and to the right of color i to be vertical rhombi. But then we can retiling this subregion by moving all horizontal tiles to the

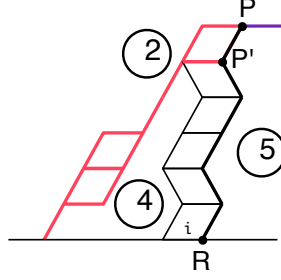


Figure 11: Triangle i cannot be in region 4.

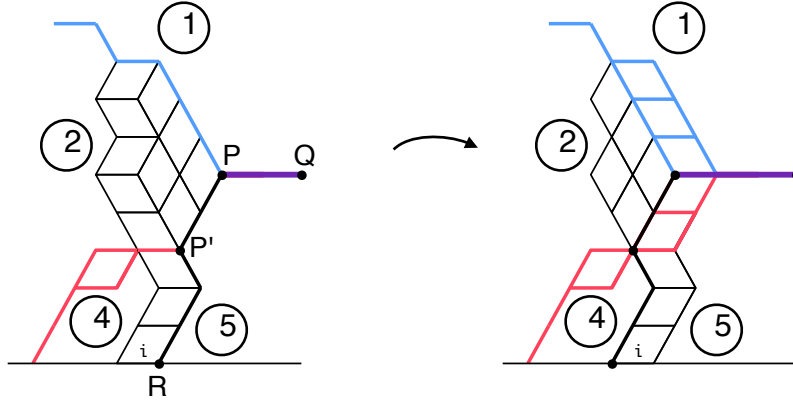


Figure 12: Triangle i cannot be in region 1.

right end of region 2 and shifting all vertical rhombi one unit to the left. This results in a new tiling of the triangle of side length $n - 1$ which has the same system of permutations, but where the length of PQ is larger, a contradiction.

It follows that triangle i is in region 1 as desired. This concludes the proof. \square

4.2 Acyclic systems of permutations and triangle positions

The following is the main result of this section.

Theorem 4.3 (Acyclic systems of permutations and triangle positions). *In a lozenge tiling of a triangle, the acyclic system of permutations determines uniquely the numbered positions of the unit triangles. Conversely, the numbered positions of the triangles and one permutation of the system determine uniquely the other two permutations.*

4.2.1 A permutation factorization

We begin by introducing a way of factoring a permutation uniquely into a particular standard form. This factorization will play an important role in our analysis of lozenge tilings.

Lemma 4.4. *Every permutation u of $[n]$ can be written uniquely in the form*

$$u = (n, \dots, p_n) \circ \dots \circ (2, 3, \dots, p_2) \circ (1, 2, \dots, p_1)$$

for integers p_1, \dots, p_n such that $i \leq p_i \leq n$ for all i .

Proof. We proceed by induction on n . The case $n = 1$ is trivial. Consider a permutation π of $[n]$. For the equation to be true, we must have $u^{-1}(1) = p_1$, so p_1 is determined by u . Then we see that $u \circ (p_1, \dots, 2, 1)$ leaves 1 fixed, and can be regarded as a permutation of $[2, \dots, n]$. By the induction hypothesis, it can be written uniquely as

$$u \circ (p_1, \dots, 2, 1) = (n, \dots, p_n) \circ (n-1, \dots, p_{n-1}) \circ \dots \circ (2, \dots, p_2),$$

for $i \leq p_i \leq n$. This gives the unique such expression for u . □

Lemma 4.5. *Similarly, every permutation v of $[n]$ can be written uniquely in the form*

$$v = (1, \dots, q_1) \circ \dots \circ (n-1, n-2, \dots, q_{n-1}) \circ (n, n-1, \dots, q_n).$$

for integers q_1, \dots, q_n such that $i \geq q_i \geq 1$ for all i .

The following lemma tells us how to compute the values of p_1, \dots, p_n and q_1, \dots, q_n in terms of the permutations u and v .

Lemma 4.6. *In the two lemmas above we have*

$$\begin{aligned} p_k &= k + |\{\ell > k : u^{-1}(\ell) < u^{-1}(k)\}| \\ q_k &= k - |\{\ell < k : v^{-1}(\ell) > v^{-1}(k)\}| \end{aligned}$$

Proof. We prove the result for p_k ; the proof for q_k is analogous. The inverse of u is

$$u^{-1} = (p_1, \dots, 2, 1) \circ (p_2, \dots, 3, 2) \circ \dots \circ (p_n, \dots, n).$$

Let $\pi_k := (p_k, \dots, k) \circ (p_{k+1}, \dots, k+1) \circ \dots \circ (p_n, \dots, n)$, a permutation of $\{k, \dots, n\}$. Note that, to obtain π_k from π_{k+1} (which is a permutation of $\{k+1, \dots, n\}$), we simply insert p_k at the beginning of the permutation, and subtract 1 from all entries less than or equal to it. For instance, if $(p_1, \dots, p_6) = (3, 2, 4, 5, 6, 6)$, then the permutations π_6, \dots, π_1 are 6, 65, 564, 4563, 24563, 314562, respectively. In particular, the relative order of $\pi_{k+1}(a)$ and $\pi_{k+1}(b)$ (where $a, b \geq k+1$) is preserved in π_k .

It follows that, for $\ell > k$, we have $u^{-1}(\ell) = \pi_1(\ell) < \pi_1(k) = u^{-1}(k)$ if and only if $\pi_k(\ell) < \pi_k(k)$. But $\pi_k(k) = p_k$, so

$$|\{\ell > k : u^{-1}(\ell) < u^{-1}(k)\}| = |\{\ell > k : \pi_k(\ell) < \pi_k(k)\}| = p_k - k,$$

as desired. □

4.2.2 Tilings and wiring diagrams

The possible positions of the triangles in a lozenge tiling of $n\Delta_2$ naturally correspond to triples in the triangular array of non-negative natural numbers (x_1, x_2, x_3) whose sum is equal to $n - 1$. The unit triangles at the corners A, B , and C have coordinates $(n - 1, 0, 0)$, $(0, n - 1, 0)$, and $(0, 0, n - 1)$, respectively. We denote by G_n the directed graph whose vertices are the triples in this triangular array, and where each node which is not in the bottom row is connected to the two nodes directly below it. There is a natural bijection between the lozenge tilings of $n\Delta_2$ and the vertex-disjoint routings to the bottom n vertices of the graph G_n : simply place one rhombus over each edge in the routing, one vertical rhombus over each isolated vertex, and one triangle over the top vertex of each path in the routing [12]. See Figure 13 for an example.

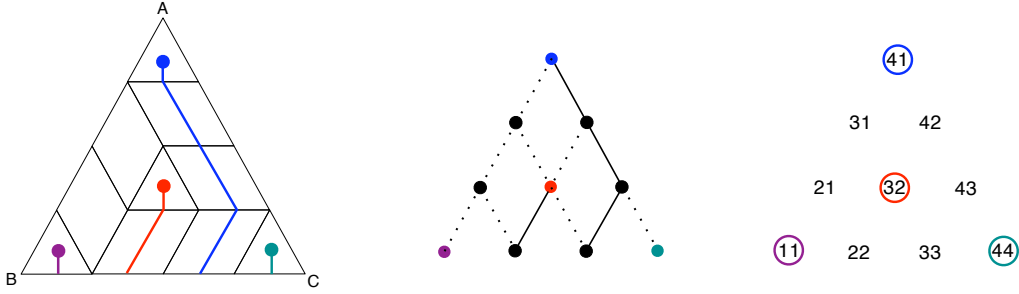


Figure 13: Left: A tiling of $4\Delta_2$. Middle: The corresponding routing of G_4 . Right: The coordinates (p, q) of the vertices of G_4 .

We perform a change of coordinates and label the nodes of G_n with pairs of numbers (p, q) , where $p = x_1 + x_3 + 1$ and $q = x_3 + 1$. The p and q coordinates range from 1 to n , and increase in the northeast and southeast directions, respectively. Figure 13 shows the coordinates (p, q) of the graph G_4 . Given a lozenge tiling T , and the corresponding routing of G_n , number the positions of the unit triangles from 1 up to n , such that the vertex of the i th triangle is routed to the vertex with (p, q) -coordinate equal to (i, i) . Let (p_i, q_i) be the position of the i th triangle in T . Since the triangles of T are spread out, we have $1 \leq q_i \leq i \leq p_i \leq n$ for all i . We now show that knowing the positions (p_i, q_i) is equivalent to knowing the system of permutations (u, v, w) .

Lemma 4.7. *The ordered list of positions (p_i, q_i) of the triangles in a lozenge tiling determines the permutations u and v as follows:*

$$\begin{aligned} u &= (n, \dots, p_n) \circ \dots \circ (2, \dots, p_2) \circ (1, \dots, p_1) \\ v &= (1, \dots, q_1) \circ \dots \circ (n - 1, \dots, q_{n-1}) \circ (n, \dots, q_n). \\ w &= n \dots 321 \end{aligned}$$

Proof. The equation $w = n \dots 321$ holds by assumption. We prove the formula for u ; the proof for v is analogous. The color i splits naturally into three broken rays $R_A(i)$, $R_B(i)$, and $R_C(i)$ centered at the i th triangle and pointing away from vertices A , B , and C respectively. Consider the pseudolines $L(i) = R_A(i) \cup R_C(i)$ for $1 \leq i \leq n$. We can regard this pseudoline arrangement as a wiring diagram for the permutation u .

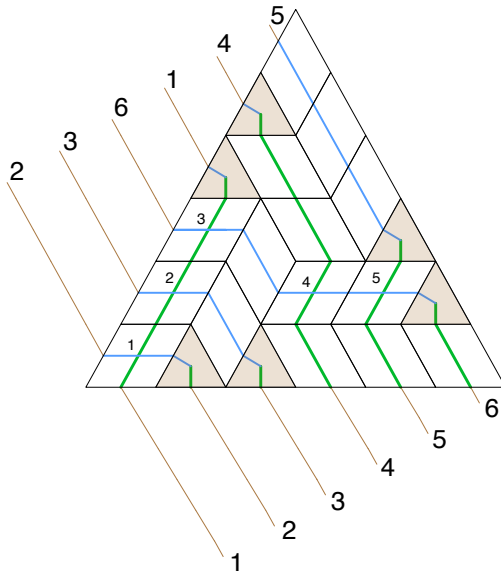


Figure 14: A tiling with $p = (4, 2, 3, 5, 6, 6)$. We have $u = 236145 = ()(s_5)(s_4)()()(s_1 s_2 s_3) = (\mathbf{6})(\mathbf{56})(\mathbf{45})(\mathbf{3})(\mathbf{2})(1234)$. In the other direction we have $q = (1, 2, 3, 1, 4, 5)$ and $v = 412563 = ()()() (s_3 s_2 s_1)(s_4)(s_5) = (\mathbf{11})(\mathbf{22})(\mathbf{33})(\mathbf{4321})(\mathbf{54})(\mathbf{65})$.

To express u as a product of transpositions, it suffices to linearly order the crossings from bottom to top, in an order compatible with the partial order given by the wiring diagram, and multiply them left to right. One way of doing it is to proceed up the ray $R_A(n)$, then up the ray $R_A(n-1)$, and so on up to the ray $R_A(1)$, recording every crossing that we see along the way. This procedure lists every crossing exactly once, and the crossings along ray $R_A(i)$ correspond to the transpositions $s_i, s_{i+1}, \dots, s_{p_i-1}$ (where $s_j = (j, j+1)$) which multiply to the cycle (i, \dots, p_i) . \square

Notice that in Lemma 4.7, we need the **ordered** list of positions of the triangles to determine the system of permutations. Figure 15 illustrates this.

We have now done all the work to prove the main result of this section.

Proof of Theorem 4.3. If we are given the numbered triangle positions and a permutation of the system, we can assume without loss of generality that the given permutation is $w = n \dots 1$. Lemma 4.7 then tells us how to obtain the two remaining permutations u

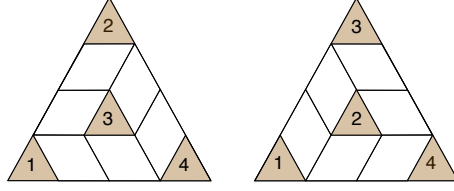


Figure 15: Two tilings with the same unordered set of triangles and different systems of permutations.

and v . Moreover, given that $1 \leq q_i \leq i \leq p_i \leq n$, Lemmas 4.4 and 4.5 imply that this procedure is reversible, and Lemma 4.6 gives us an explicit way of computing the triangle positions in terms of u and v . \square

4.2.3 From acyclic systems to triangle positions: another description

Let T be a lozenge tiling of $n\Delta_2$ and σ be its corresponding acyclic system of permutations. In addition to Lemmas 4.6 4.7, we now present a different way of computing the triangle positions of T in terms of σ .

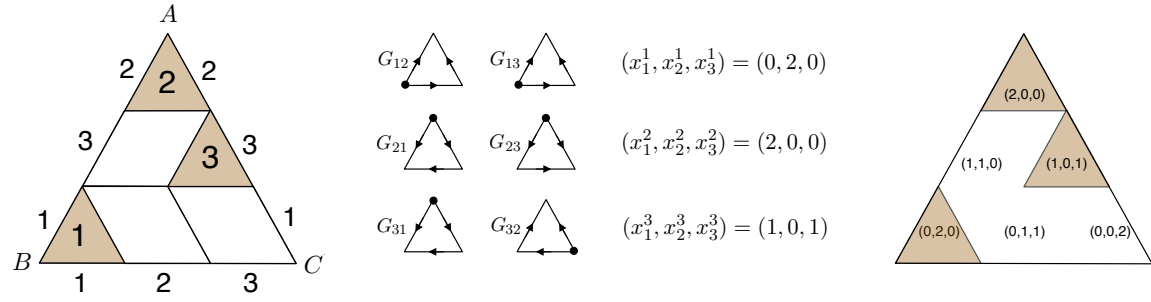


Figure 16: Obtaining the triangle positions from the system of permutations.

As before, we identify the positions of the unit triangles in T with triples in the triangular array of non-negative natural numbers (x_1, x_2, x_3) whose sum is equal to $n - 1$. For $1 \leq i \neq j \leq n$ define the *directed graph* G_{ij} on the triangle ABC ; we orient edge e from i to j , according to the order in which i and j appear on e in the system of permutations σ .

Since the system of permutations is acyclic, each graph G_{ij} is acyclic and has a unique source. The position (x_1^i, x_2^i, x_3^i) of the i -th triangle is given by:

$$\begin{aligned} x_1^i &= |\{j \neq i : A \text{ is the unique source of } G_{ij}\}|, \\ x_2^i &= |\{j \neq i : B \text{ is the unique source of } G_{ij}\}|, \\ x_3^i &= |\{j \neq i : C \text{ is the unique source of } G_{ij}\}|. \end{aligned}$$

We illustrate this in an example in Figure 16. This result is proved in greater generality in Section 7.

5 The Acyclic System Conjecture

The main goal of this section is to introduce the concept of the system of permutations of a higher dimensional subdivision of a simplex. We prove that the system of permutations of a subdivision is acyclic, and conjecture that the converse holds as well.

Let S be a subdivision on $n\Delta_{d-1}$. As mentioned in Remark 2.2, in order to prevent confusion we will denote indices in the set $[n]$ by the letters i, j, k, ℓ , and indices in the set $[d]$ by the letters a, b . We also denote the vertices of the simplex Δ_{d-1} by w_1, \dots, w_d , the vertices of $n\Delta_{d-1}$ by nw_1, \dots, nw_d , the vertices of Δ_{n-1} by v_1, \dots, v_n , and the vertices of $d\Delta_{n-1}$ by dv_1, \dots, dv_n .

The restriction $S|_{nw_aw_b}$ of the subdivision S to the edge nw_aw_b is the subdivision of the segment nw_aw_b given by the Minkowski sums $S_1 + \dots + S_n \in S$ for which $S_i \subset \{w_a, w_b, w_aw_b\}$ for all $i = 1, \dots, n$.

Definition 5.1 (The permutation of an edge). Since $S|_{nw_aw_b}$ is a subdivision, for each $i \in [n]$ there is a unique cell having $i - 1$ summands equal to w_b , $n - i$ summands equal to w_a , and one summand (which we denote $S_{\sigma_{ab}(i)}$) equal to w_aw_b . It is easy to see that σ_{ab} is a permutation of $[n]$, which we call *the permutation of the edge nw_aw_b* . (Note that σ_{ab} is the reverse of σ_{ba} for any $1 \leq a \neq b \leq d$.)

Remark 5.2. It is worth describing more explicitly the subdivision along each edge. As we traverse the edge nw_aw_b from the vertex nw_a to nw_b , the first edge of S that we encounter has the form $w_a + \dots + w_a + w_aw_b + w_a + \dots + w_a$. Each subsequent edge is obtained from the previous one by converting the summand w_aw_b into w_b and converting one of the summands w_a into w_aw_b . The permutation σ_{ab} tells us the order in which the summands w_a are converted to w_aw_b (and then to w_b).

Definition 5.3 (The system of permutations). The *system of permutations of a fine mixed subdivision S* of $n\Delta_{d-1}$ is the collection $\sigma(S) = (\sigma_{ab})_{1 \leq a \neq b \leq d}$ of permutations σ_{ab} of the edges nw_aw_b .

Example 5.4. Consider the subdivision S of $3\Delta_2$ given on the left hand side of Figure 7, with the small difference that we now call the vertices $3w_1, 3w_2$, and $3w_3$ instead of A, B , and C . Writing only the one dimensional cells of the subdivision restricted to the edges of the triangles we have:

$$\begin{aligned} S|_{3w_1w_2} &= S|_{3w_2w_1} = \{w_1w_2 + w_1 + w_1, w_2 + w_1 + w_1w_2, w_2 + w_1w_2 + w_2\} \\ S|_{3w_2w_3} &= S|_{3w_3w_2} = \{w_2 + w_2w_3 + w_2, w_2w_3 + w_3 + w_2, w_3 + w_3 + w_2w_3\} \end{aligned}$$

$$S|_{3w_3w_1} = S|_{3w_1w_3} = \{w_3 + w_1w_3 + w_3, w_3 + w_1 + w_1w_3, w_1w_3 + w_1 + w_1.\}$$

The system of permutations $\sigma(S) = (\sigma_{ab})_{1 \leq a \neq b \leq 3}$ is then given by

$$\begin{array}{ll} \sigma_{12} = 132 & \sigma_{21} = 231 \\ \sigma_{23} = 213 & \sigma_{32} = 312 \\ \sigma_{31} = 231 & \sigma_{13} = 132. \end{array}$$

Notice that this system coincides with the restriction of the coloring of S to the edges of the triangle.

Definition 5.5. A *system of permutations* on the edges of $n\Delta_{d-1}$ is a collection $\sigma = (\sigma_{ab})_{1 \leq a \neq b \leq d}$ of permutations σ_{ab} of $[n]$ such that σ_{ab} is the reverse of σ_{ba} for all a, b . For each pair $1 \leq i \neq j \leq n$ we define the *directed graph* $G_{ij}(\sigma)$ of σ as the complete graph on $[d]$, where edge ab is directed $a \rightarrow b$ if and only if the permutation σ_{ab} is of the form $\dots i \dots j \dots$. We say that a system of permutations σ is *acyclic* if and only if all the graphs $G_{ij}(\sigma)$ are acyclic.

In other words, a system of permutations on the edges of a simplex is acyclic if and only if there is no closed walk along the edges such that the permutation on every directed edge of the walk has the form $\dots i \dots j \dots$ for some i and j .

Theorem 5.6. *Let S be a fine mixed subdivision of $n\Delta_{d-1}$, and $\sigma(S)$ be the corresponding system of permutations. Then $\sigma(S)$ is acyclic.*

Proof. The cases $d = 1, 2$ are trivial. The case $d = 3$ was shown in Theorem 4.2. For $d > 3$, notice that an orientation of the complete graph K_d is acyclic if and only if every triangle $w_a w_b w_c$ is acyclic. But the orientation of triangle $w_a w_b w_c$ is given by the subdivision $S|_{nw_a w_b w_c}$, and so it is acyclic by Theorem 4.2. \square

We conjecture that the converse also holds:

Acyclic System Conjecture 5.7. *Any acyclic system of permutations on the edges of the simplex $n\Delta_{d-1}$ is achievable as the system of permutations of a fine mixed subdivision.*

Theorem 4.2 says that the Acyclic System Conjecture 5.7 is true for $d = 3$.

6 Duality, deletion and contraction

Before we continue extending the results of Section 4 from two dimensions to higher dimensions we need some general machinery. This section introduces the notion of duality, deletion and contraction for acyclic systems of permutations on the edges of a simplex. We show that our definitions are compatible with the previously known notions of duality, deletion and contraction for subdivisions [3, 18].

6.1 Duality for subdivisions

We introduce a notion of duality between subdivisions of $n\Delta_{d-1}$ and subdivisions of $d\Delta_{n-1}$. We begin by recalling the one-to-one correspondence between subdivisions of $n\Delta_{d-1}$ and triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$. This equivalent point of view has the drawback of bringing us to a higher-dimensional picture. Its advantage is that it simplifies greatly the combinatorics of the tiles, which are now just simplices.

Let v_1, \dots, v_n and w_1, \dots, w_d be the vertices of Δ_{n-1} and Δ_{d-1} , so that the vertices of $\Delta_{n-1} \times \Delta_{d-1}$ are of the form $v_i \times w_e$. A triangulation T of $\Delta_{n-1} \times \Delta_{d-1}$ is given by a collection of simplices. For each simplex t in T , consider the fine mixed cell whose i -th summand is $w_a w_b \dots w_c$, where a, b, \dots, c are the indices $e \in [d]$ such that $v_i \times w_e$ is a vertex of t . These fine mixed cells constitute the subdivision of $n\Delta_{d-1}$ corresponding to T . This bijection is a special case of the more general Cayley trick; for a more thorough discussion see [18]. Figure 17 shows an example of a triangulation of the triangular prism $\Delta_1 \times \Delta_2 = 12 \times ABC$, and the corresponding subdivision of $2\Delta_2$, whose three tiles are $ABC + B$, $AC + AB$, and $C + ABC$.

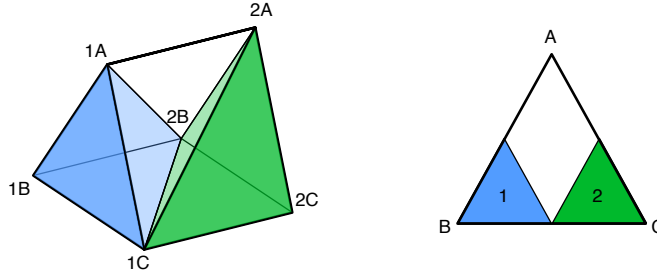


Figure 17: The Cayley trick

Since $\Delta_{n-1} \times \Delta_{d-1} \cong \Delta_{d-1} \times \Delta_{n-1}$, the previous bijection gives us a notion of duality between the subdivisions of $n\Delta_{d-1}$ and the subdivisions of $d\Delta_{n-1}$. Given a subdivision S of $n\Delta_{d-1}$, we denote by S^* the *dual subdivision* of $d\Delta_{n-1}$ that corresponds to the triangulation of $\Delta_{d-1} \times \Delta_{n-1}$. More explicitly, the dual of a mixed cell $S_1 + \dots + S_n$ in S is the mixed cell $Z_1 + \dots + Z_d$ in S^* , where $Z_a = \{v_i : w_a \in S_i\}$. Figure 18 shows an example of a subdivision of $3\Delta_{4-1}$, its dual subdivision of $4\Delta_{3-1}$ and the Minkowski sum decompositions of the full dimensional cells.

6.2 Duality for acyclic systems of permutations

Let $\sigma = (\sigma_{ab})_{1 \leq a \neq b \leq d}$ be an acyclic system of permutations on the edges of the simplex $n\Delta_{d-1}$. Recall that, for each pair $1 \leq i \neq j \leq n$, the graph $G_{ij}(\sigma)$ on $[d]$ vertices has a directed edge $a \rightarrow b$ if and only if the permutation σ_{ab} is of the form $\dots i \dots j \dots$. Since this graph is acyclic and complete, it can be naturally regarded as a permutation σ_{ij}^* of $[d]$.

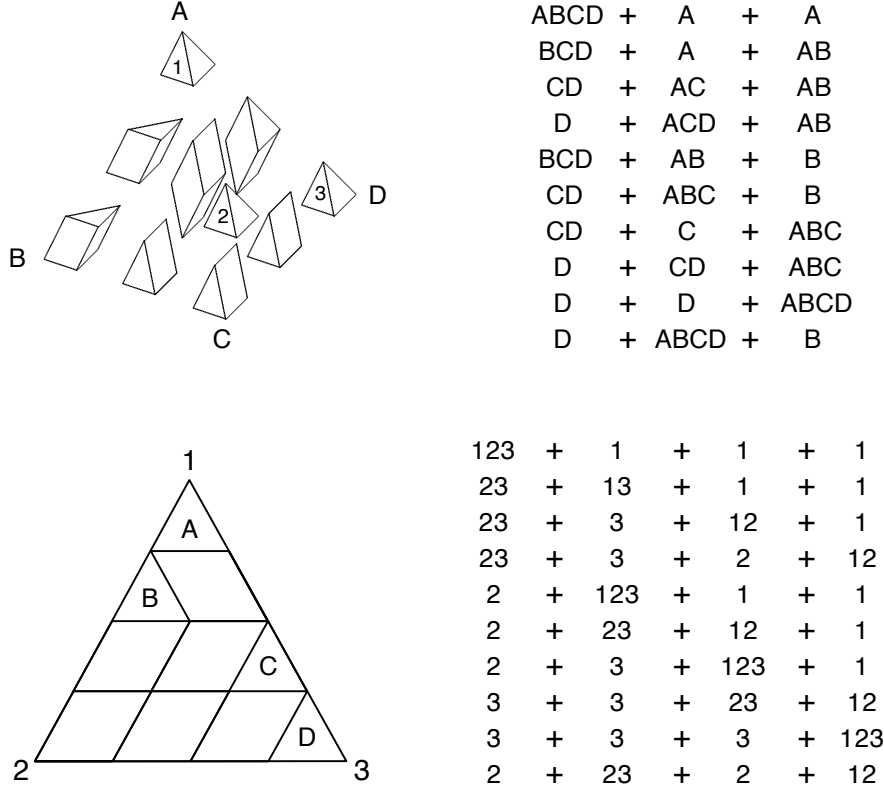


Figure 18: A subdivision S of $3\Delta_{4-1}$, its dual subdivision S^* of $4\Delta_{3-1}$ and the Minkowski sum decompositions of the full dimensional cells. The systems of permutations $\sigma = \sigma(S)$ and $\sigma^* = \sigma(S^*)$ are given by $\sigma_{AB} = 132$, $\sigma_{AC} = 123$, $\sigma_{AD} = 123$, $\sigma_{BC} = 123$, $\sigma_{BD} = 123$, $\sigma_{CD} = 123$, and $\sigma_{12}^* = ABCD$, $\sigma_{23}^* = BACD$, $\sigma_{31}^* = DCBA$.

More precisely, for $a \in [d]$, $\sigma_{ij}^*(a)$ is the vertex of $G_{ij}(\sigma)$ whose out-degree is equal to $d - a$.

Definition 6.1. The *dual system* of an acyclic system σ of $n\Delta_{d-1}$ is the system of permutations $\sigma^* = (\sigma_{ij}^*)_{1 \leq i \neq j \leq n}$ on the edges of $d\Delta_{n-1}$.

Example 6.2. If $\sigma_{12} = 132$, $\sigma_{23} = 213$, $\sigma_{31} = 231$, then $\sigma_{12}^* = 132$, $\sigma_{23}^* = 231$, $\sigma_{31}^* = 321$.

Lemma 6.3. The permutation σ_{ab} is of the form $\dots i \dots j \dots$ if and only if the permutation σ_{ij}^* is of the form $\dots a \dots b \dots$.

Proof. The permutation σ_{ab} is of the form $\dots i \dots j \dots$ if and only if the graph $G_{ij}(\sigma)$ has a directed edge pointing from a to b , which is the case if and only if the permutation σ_{ij}^* is of the form $\dots a \dots b \dots$. \square

Proposition 6.4. The system of permutations σ^* is acyclic and $(\sigma^*)^* = \sigma$.

Proof. Suppose σ^* has a cycle $\dots a \dots b \dots, \dots, \dots a \dots b \dots$ passing through the vertices i_1, \dots, i_r of $d\Delta_{n-1}$. Then the permutation σ_{ab} is of the form $\dots i_1 \dots i_2 \dots i_r \dots i_1 \dots$ which is a contradiction. Therefore the system σ^* is acyclic and its dual system $(\sigma^*)^*$ is well defined.

Let $\alpha = (\sigma^*)^*$ and α_{ab} be the permutation of $(\sigma^*)^*$ on the edge $nw_a w_b$ of $n\Delta_{d-1}$. The permutation σ_{ab} is of the form $\dots i \dots j \dots$ in the system σ if and only if the permutation σ_{ij}^* is of the form $\dots a \dots b \dots$ in the system σ^* , which is true if and only if α_{ab} is of the form $\dots i \dots j \dots$ in the system $(\sigma^*)^*$. Then $\sigma_{ab} = \alpha_{ab}$ which implies that $\sigma = (\sigma^*)^*$. \square

6.3 Deletion and contraction for subdivisions

Recall that we denote the vertices of Δ_{d-1} by w_1, \dots, w_d , and the vertices of Δ_{n-1} by v_1, \dots, v_n .

Definition 6.5. Let S be a subdivision of $n\Delta_{d-1}$. For $i \in [n]$, the *deletion* $S \setminus i$ is the subdivision of $(n-1)\Delta_{d-1}$ whose mixed cells correspond to the Minkowski sums obtained from the Minkowski sums of S by deleting the i th summand.

We can also think of the subdivision $S \setminus i$ in terms of the Cayley trick. The subdivision S corresponds to a triangulation T of $\Delta_{n-1} \times \Delta_{d-1}$; The subdivision $S \setminus i$ corresponds to the restriction of T to the facet $\Delta_{n-2} \times \Delta_{d-1}$, where Δ_{n-2} represents the simplex with vertices $v_1, \dots, \hat{v}_i, \dots, v_n$. From this description it follows that $S \setminus i$ is indeed a subdivision of $(n-1)\Delta_{d-1}$. The subdivision $S \setminus i$ can be also described in terms of the coloring of S : it is the result of deleting the mixed cells of S that intersect the color i , and gluing the remaining pieces together.

Definition 6.6. Let S be a subdivision of the simplex $n\Delta_{d-1}$. For $a \in [d]$, the *contraction* S/a is the subdivision of $n\Delta_{d-2}$ which consists of the cells $S_1 + \dots + S_n$ of S such that $S_i \subset \{w_1, \dots, \hat{w}_a, \dots, w_d\}$.

In other words, the contraction S/a is the restriction of the subdivision S to the facet of $n\Delta_{d-1}$ that does not contain the vertex nw_a . It also corresponds to the restriction of the triangulation T to the facet $\Delta_{n-1} \times \Delta_{d-2}$, where Δ_{d-2} is the simplex with vertices $w_1, \dots, \hat{w}_a, \dots, w_d$. From this description it follows that S/a is indeed a subdivision of $n\Delta_{d-2}$.

We call these operations “deletion” and “contraction” because they resemble the analogous notions in matroid theory, and have several of the same properties. See [3] for their counterparts in tropical oriented matroids.

6.4 Deletion and contraction for systems of permutations

Definition 6.7. Let σ be a system of permutations on the edges of $n\Delta_{d-1}$. For $i \in [n]$, the *deletion* $\sigma \setminus i$ is a system of permutations on the edges of $(n-1)\Delta_{d-1}$, obtained from σ by deleting the number i from each permutation.

For example, for the system of permutations $\sigma = \{1423, 3124, 4321\}$ on the edges of $4\Delta_3$ in Figure 5, we get $\sigma \setminus 2 = \{143, 314, 431\}$. Note that $\sigma \setminus 2$ is the system of permutations of the deletion $S \setminus 2$ of the subdivision S in the same figure.

Proposition 6.8. *If $\sigma = \sigma(S)$ is the system of permutations of a subdivision S , then $\sigma \setminus i = \sigma(S \setminus i)$.*

Proof. Let S be a subdivision of $n\Delta_{d-1}$. Recall that the permutation σ_{ab} is defined as the permutation of the subdivision $S|_{nw_a w_b}$, which is the restriction of S to the edge $nw_a w_b$. The subdivision $(S \setminus i)|_{nw_a w_b}$ is obtained from $S|_{nw_a w_b}$ by deleting the one dimensional cell whose i th component is $w_a w_b$. Therefore, the permutation $\sigma_{ab}(S \setminus i)$ is obtained from the permutation σ_{ab} by deleting the number i . \square

Definition 6.9. Let σ be a system of permutations on the edges of $n\Delta_{d-1}$. For $a \in [d]$, the *contraction* σ/a is the restriction of σ to the edges of the facet of $n\Delta_{d-1}$ that do not contain the vertex nw_a .

Proposition 6.10. *If $\sigma = \sigma(S)$ is the system of permutations of a subdivision S , then $\sigma/a = \sigma(S/a)$.*

Proof. This is clear from the definitions. \square

6.5 Properties

In this subsection we show that the operations of deletion and contraction are dual to each other, and that the dual system of the system of permutations of a subdivision S is equal to the system of permutations of the dual subdivision S^* . This result will be a key lemma in Section 8.

Proposition 6.11. *Let S be a subdivision of $n\Delta_{d-1}$, and let $i \in [n]$, $a \in [d]$. Then:*

1. $(S \setminus i)^* = S^* / i$
2. $(S/a)^* = S^* \setminus a$

Proposition 6.12. *Let σ be an acyclic system of permutations of $n\Delta_{d-1}$, and let $i \in [n]$, $a \in [d]$. Then:*

1. $(\sigma \setminus i)^* = \sigma^* / i$
2. $(\sigma/a)^* = \sigma^* \setminus a$

The geometric content of Propositions 6.11 and 6.12 is the following: The deletion of a color i in a subdivision S corresponds to the contraction of the vertex i in the dual subdivision S^* , and viceversa. Proposition 6.11 follows directly from the Cayley trick, and Proposition 6.12 follows from the definitions.

Proposition 6.13. *Let S be a subdivision of $n\Delta_{d-1}$ and $\sigma(S)$ be the associated system of permutations. Then $\sigma(S)^* = \sigma(S^*)$.*

Proof. First we reduce the proposition to the case $n = 2$. Let S be a subdivision of $n\Delta_{d-1}$, $\sigma = \sigma(S)$ and $\pi = \sigma(S^*)$. We need to show $\sigma_{ij}^* = \pi_{ij}$ for all $1 \leq i \neq j \leq n$. On one hand we have $G_{ij}(\sigma) = G_{ij}(\sigma(S \setminus \{1 \dots \hat{i} \dots \hat{j} \dots n\}))$. On the other hand, the permutation $\pi_{ij} = \sigma(S^* / \{1 \dots \hat{i} \dots \hat{j} \dots n\}) = \sigma((S \setminus \{1 \dots \hat{i} \dots \hat{j} \dots n\})^*)$. Therefore the result is equivalent to proving that the permutation σ_{ij}^* associated to graph $G_{ij}(\sigma(\tilde{S}))$, is equal to the permutation π_{ij} of the reduced system $\pi = \sigma(\tilde{S}^*)$, where $\tilde{S} = S \setminus \{1 \dots \hat{i} \dots \hat{j} \dots n\}$. Since \tilde{S} is a subdivision of $2\Delta_{d-1}$, This last statement is equivalent to proving the proposition for subdivisions of $2\Delta_{d-1}$.

It is not hard to see that the full-dimensional cells of any subdivisions of $2\Delta_{d-1}$ are of the form

$$\begin{array}{rcl}
 & v_1 & v_2 \\
 w_{a_1} w_{a_2} w_{a_3} \dots w_{a_d} & + & w_{a_1} \\
 w_{a_2} w_{a_3} \dots w_{a_d} & + & w_{a_1} w_{a_2} \\
 w_{a_3} \dots w_{a_d} & + & w_{a_1} w_{a_2} w_{a_3} \\
 \vdots & & \vdots \\
 w_{a_d} & + & w_{a_1} w_{a_2} w_{a_3} \dots w_{a_d}
 \end{array}$$

where a_1, \dots, a_d is a permutation of $[d]$. In this case, the graph G_{12} has a directed edge $a_k \rightarrow a_{k'}$ if and only if $k < k'$, and therefore $\sigma(S)^* = a_1, \dots, a_d$. On the other hand, the dual subdivision S^* of $d\Delta_1$ is given by:

$$\begin{array}{cccccc}
 w_{a_1} & & w_{a_2} & & w_{a_3} & & \dots & & w_{a_d} \\
 12 & + & 1 & + & 1 & + & \dots & + & 1 \\
 2 & + & 12 & + & 1 & + & \dots & + & 1 \\
 2 & + & 2 & + & 12 & + & \dots & + & 1 \\
 \vdots & & \vdots & & \vdots & & & & \vdots \\
 2 & + & 2 & + & 2 & + & \dots & + & 12
 \end{array}$$

Therefore $\sigma(S)^* = \sigma(S^*) = a_1 \dots a_d$ as we wished to show. \square

The operations of restriction, contraction, and duality, and their elegant properties, turn out to be a useful tool towards the study of mixed subdivisions. The following section is a clear illustration of this.

7 From systems of permutations to simplex positions

While a subdivision of $n\Delta_{d-1}$ is not uniquely determined by its system of permutations, in this section we will see that the positions of its simplices **are** completely determined. We already proved this result for lozenge tilings in Theorem 4.3. We now prove the result in general.

Theorem 7.1 (Acyclic systems of permutations and simplex positions). *The numbered positions of the simplices in a fine mixed subdivision S of $n\Delta_{d-1}$ are completely determined by its system of permutations $\sigma = \sigma(S)$. More precisely, the Minkowski decomposition of the i th simplex is*

$$w_{a_1} + \cdots + w_{a_{i-1}} + w_1 w_2 \cdots w_n + w_{a_{i+1}} + \cdots + w_{a_n}$$

where a_j is the unique source of the acyclic graph $G_{ij}(\sigma)$ for all $j \neq i$.

Proof. The Minkowski sum decomposition of the i th simplex T in the subdivision S has i th component equal to $w_1 \cdots w_d$. This implies that the letter v_i appears in all the components of the Minkowski decomposition of the dual cell T^* . It follows that T^* is the unique full dimensional cell of the subdivision S^* that contains the vertex dv_i of $d\Delta_{n-1}$. This dual cell T^* is completely determined by the colors of the edges adjacent to the vertex dv_i in S^* , which are precisely the sources of $G_{ij}(\sigma)$ with $j \neq i$. More explicitly, for each index $a \in [d]$, the Minkowski decomposition of the dual cell T^* has a th component equal to $\{v_j, j \in [n] : j = i \text{ or } \text{source}(G_{ij}) = a\}$. Dualizing, we get the desired Minkowski decomposition of the simplex T . \square

Figure 19 shows two examples of how to compute the positions of the simplices in a subdivision from its system of permutations. For instance, the system of permutations of the top subdivision of $3\Delta_2$ gives rise to three acyclic graphs G_{12}, G_{13} and G_{23} . The source of G_{12} is equal to C , and the source of G_{13} is equal to B . Therefore simplex 1 has Minkowski sum given by $ABC + C + B$.

Remark 7.2. In dimension 2, our proofs of Theorem 4.3 and Theorem 7.1 give two descriptions of the triangle positions as a function of the acyclic system of permutations. The first description, and Lemma 4.6 in particular, gives a simple computation of these positions. The second description gives us additional information about the triangles, namely, their Minkowski sum decompositions.

For higher d , this second computation works without modification. It would also be interesting to generalize the first one; *i.e.*, to find a direct description of the positions of the simplices in the spirit of our proof of Theorem 4.3.

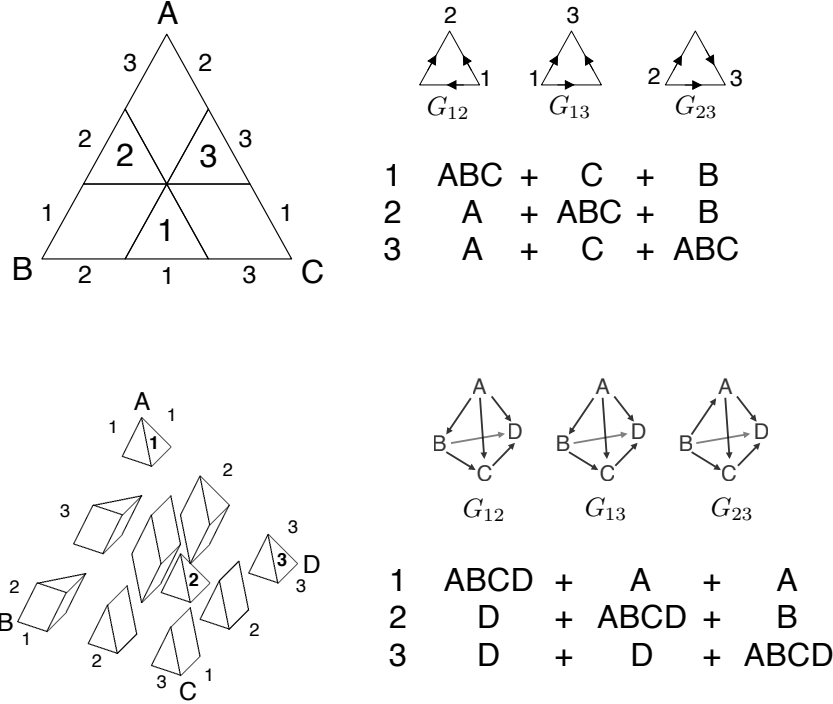


Figure 19: How to obtain the positions (and Minkowski sum decompositions) of the simplices from the system of permutations.

8 The Spread Out Simplices Conjecture

One of the motivations of this paper is the Spread Out Simplices Conjecture of Ardila and Billey. They showed that every subdivision of $n\Delta_{d-1}$ contains precisely n unit simplices, and studied where those simplices could be located. They conjectured that the possible positions of the simplices are given by the bases of the matroid determined by the lines in a generic complete flag arrangement. For details, see [2].

The following is an equivalent statement. Recall that a collection of n simplices in $n\Delta_{d-1}$ is said to be *spread out* if no subsimplex of size k contains more than k of them.

Spread Out Simplices Conjecture 8.1. [2, Conjecture 7.1]. *A collection of n simplices in $n\Delta_{d-1}$ can be extended to a fine mixed subdivision if and only if it is spread out.*

8.1 Regular subdivisions: Yoo's example

Question 8.3 in [2] asked whether Conjecture 8.1 is true in the more restrictive context of **regular** subdivisions. Similarly, one may ask the same question for Conjecture 5.7. Hwanchul Yoo [20] showed that these statements are false in that context, even for $d = 3$.

Figure 20 shows an acyclic system of permutations on $6\Delta_2$ and a collection of 6 triangles which can only be realized by two subdivisions, neither of which is regular.

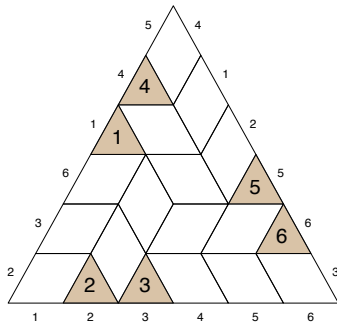


Figure 20: A non-regular subdivision of $6\Delta_2$. Its system of permutations and triangle positions cannot be achieved by a regular subdivision. (Example by Hwanchul Yoo).

8.2 The simplex positions are spread out

We identify the possible positions of the simplices in a subdivision with the lattice points of the simplex

$$\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 + \dots + x_d = n - 1 \text{ and } x_1, \dots, x_d \geq 0\}.$$

The previous theorem leads us to the following definition.

Definition 8.2. *The set of positions of the simplices $P(\sigma)$ of an acyclic system of permutations σ of $n\Delta_{d-1}$ is the following: For $1 \leq i \leq n$, the i th simplex has position (x_1^i, \dots, x_d^i) , where*

$$x_a^i = |\{j \neq i : a \text{ is the unique source of } G_{ij}(\sigma)\}|.$$

Notice that this definition makes sense for arbitrary acyclic systems, and not only for those coming from subdivisions. When σ comes from a subdivision, the set of positions $P(\sigma)$ is the one given by Theorem 7.1. (Of course, if the Acyclic System Conjecture 5.7 is true then there is no distinction here.)

Remark 8.3. As seen in the proof of Theorem 7.1, computing the positions of the simplices of σ is very easy if we know σ^* . Each simplex of σ corresponds to a vertex v of $d\Delta_{n-1}$, and its Minkowski summands are Δ_{d-1} and the $d - 1$ labels on the edges coming out of v in σ^* . For instance, if σ is the subdivision on the top of Figure 18, then its three simplices are readily given by the permutations around the dual triangle in σ^* : they are $ABCD + A + A, D + ABCD + B, D + D + ABCD$.

Recall that Ardila and Billey proved the forward direction of the Spread Out Simplices Conjecture 8.1:

Theorem 8.4 ([2, Proposition 8.2]). *The positions of the simplices of a fine mixed subdivision of $n\Delta_{d-1}$ are spread out.*

In principle, our next result generalizes this:

Theorem 8.5. *The positions of the simplices $P(\sigma)$ of an acyclic system of permutations σ of $n\Delta_{d-1}$ are spread out.*

Again, if the Acyclic System Conjecture 5.7 is true then Theorems 8.4 and 8.5 are equivalent.

In order to prove Theorem 8.5, let us define the *table of positions* $T(\sigma)$ of an acyclic system of permutations σ . This is an $n \times n$ matrix whose rows are given by the Minkowski summands of the simplices of the system of permutations σ . For example, the table of positions of the system of permutations $\{123, 231, 312\}$ on the top example in Figure 19 is

$$\begin{bmatrix} ABC & C & B \\ A & ABC & B \\ A & C & ABC \end{bmatrix}$$

For each $a \neq b \in [d]$, define the directed graph $H_{ab}(\sigma)$ as the graph on the vertex set $[n]$ containing a directed edge $i \rightarrow j$ if there is a w_a in row i and a w_b in row j which are in the same column. In the previous example, $H_{AB}(\sigma)$ is a complete graph in three vertices with directed edges $2 \rightarrow 1$, $3 \rightarrow 1$ and $3 \rightarrow 2$. Notice that in this case $H_{AB}(\sigma)$ is the complete acyclic graph that corresponds to the permutation $\sigma_{AB} = 321$. In general we have:

Lemma 8.6. *The graph $H_{ab}(\sigma)$ is a subgraph of the complete, acyclic graph on $[n]$ that corresponds to the permutation σ_{ab} .*

Proof. We need to prove that if H_{ab} has a directed edge $i \rightarrow j$ then the permutation σ_{ab} is of the form $\dots i \dots j \dots$. Suppose $i \rightarrow j$ in H_{ab} . Then there is a column ℓ of the table $T(\sigma)$ such that $w_a \in T_{i\ell}$ and $w_b \in T_{j\ell}$.

Case 1: If $\ell = j$, then $T_{ij} = w_a$. This means that the graph G_{ij} has a source at $a \in [d]$, which implies that the permutation σ_{ab} is of the form $\dots i \dots j \dots$.

Case 2: If $\ell = i$, an analogous argument works.

Case 3: If $\ell \neq i, j$, then $T_{i\ell} = w_a$ and $T_{j\ell} = w_b$, which means that the graphs $G_{i\ell}$ and $G_{j\ell}$ have sources at $a \in [d]$ and $b \in [d]$ respectively. This implies that the permutation σ_{ab} is of the form $\dots i \dots \ell \dots j \dots$. \square

Remark 8.7. The graph H_{ab} is in general only a proper subgraph of the graph that corresponds to σ_{ab} . In particular, in contrast with the 2-dimensional case, the ordered list of positions of the simplices (or even their Minkowski sum decompositions) is not sufficient

to determine the system of permutations. For instance, there are exactly two subdivisions of $2\Delta_3$ for which the Minkowski decomposition of the two unit simplices are $ABCD + A$ and $B + ABCD$. These two subdivisions have different systems of permutations. The graph H_{CD} in these cases consists of two vertices C and D without any edge.

Proof of Theorem 8.5. Let σ be an acyclic system of permutations on the edges of $n\Delta_{d-1}$, $P(\sigma)$ be the set of positions of the simplices and $T(\sigma)$ be the table of positions. Suppose that there is a sub-simplex Δ of size k containing more than k simplices of σ . This sub-simplex is given by

$$\Delta = \{x = (x_1 \dots x_d) \in \mathbb{R}^d : x_i \geq m_i \text{ and } x_1 + \dots + x_d = 1\},$$

for some non-negative integers m_1, \dots, m_d such that $m_1 + \dots + m_d = n - k$. Without loss of generality, we assume that the (more than k) simplices of σ that are contained in Δ correspond to the first rows of the table $T(\sigma)$. Each one of these rows contains (off of the diagonal) at least m_a letters w_a for all $a \in [d]$. Call such a letter *dark* if it is in the shaded rectangle in Figure 21, and *light* if it is in the white square on the upper left. This shaded rectangle has width less than $n - k$.

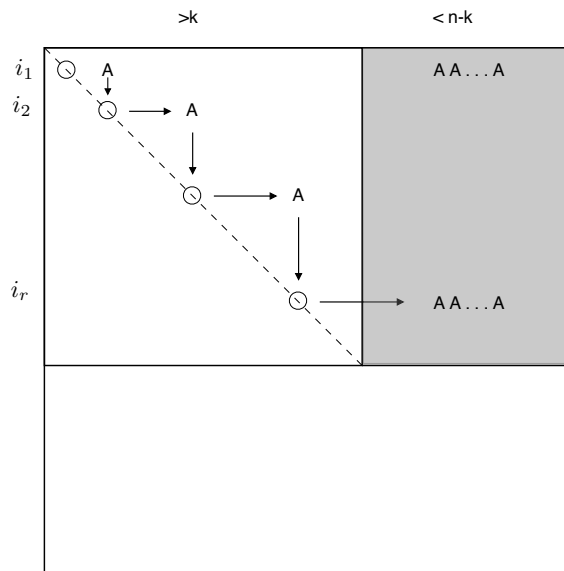


Figure 21: Table of positions $T(\sigma)$

We will prove that the first row of $T(\sigma)$ has at least m_a dark letters w_a for all $a \in [d]$. If $m_a = 0$ the result is obvious. Now suppose $m_a > 0$. If there is no light letter w_a on the first row of $T(\sigma)$ then the claim clearly follows. Otherwise, we will construct, simultaneously for all $b \in [d] \setminus \{a\}$, a path $1 = i_1 \rightarrow \dots \rightarrow i_r$ in the graphs H_{ab} ending on a row i_r that has

no light letter w_a . To do so, we start by drawing the arrows $1 = i_1 \rightarrow i_2$ in H_{ab} for all b , where i_2 is the column of the first letter w_a under consideration. If there is no light letter w_a in row i_2 , we are done; otherwise, we continue the process. Since the graphs H_{ab} have no cycles, this process must end at some row i_r .

Notice that row i_r must contain at least m_a dark letters, since it contains no light ones except for the one on the diagonal. Now consider the letters on the first row which are directly above the dark letters w_a on the row i_r . They must all be equal to w_a , or else they would form a cycle $1 = i_1 \rightarrow \dots \rightarrow i_r \rightarrow 1$ in H_{ab} for some b . Therefore, the first row of $T(\sigma)$ has at least m_a dark letters w_a as we claimed.

We finish the proof by observing that the first row contains at least m_a dark letters w_a for all $a \in [d]$, so the shaded rectangle must have width at least $m_1 + \dots + m_d = n - k$, a contradiction. \square

We conjecture that the converse of Theorem 8.5 is true as well:

Weak Spread Out Simplices Conjecture 8.8. *Any n spread out simplices in $n\Delta_{d-1}$ can be achieved as the simplices of an acyclic system of permutations.*

This conjecture is weaker than the Spread Out Simplices Conjecture 8.1, and has the advantage that it is more tractable computationally for small values of n . The Spread Out Simplices Conjecture 8.1 would follow from Conjecture 5.7 and Conjecture 8.8. In the next Section we prove these three conjectures in the special case $n = 3$.

8.3 The Spread Out Simplices Theorem for simplices of size three

Theorem 8.9 (Acyclic System Conjecture 5.7 for $n = 3$). *Every acyclic system of permutations on the edges of $3\Delta_{d-1}$ is achievable as the system of permutations of a fine mixed subdivision.*

Proof. Let σ be an acyclic system of $3\Delta_{d-1}$ and σ^* be the dual system of $d\Delta_2$. By Theorem 4.2 there exist a subdivision S^* of $d\Delta_2$ whose system of permutations is equal to σ^* . The subdivision $S = (S^*)^*$ is a subdivision of $3\Delta_{d-1}$ whose corresponding system of permutations is σ . \square

Theorem 8.10 (Weak Spread Out Simplices Conjecture 8.8 for $n = 3$). *Any three spread out simplices in $3\Delta_{d-1}$ are achievable as the simplices of an acyclic system of permutations.*

Proof. The position of a simplex of an acyclic system of permutations of $3\Delta_{d-1}$ corresponds to a Minkowski sum of the form $w_1 \dots w_d + w_{a_1} + w_{a_2}$. We identify the position of such a simplex with the pair of letters $w_{a_1} w_{a_2}$. For simplicity, we denote the letters w_1, \dots, w_d by A, B, \dots, H .

Figure 22 lists all the possible (combinatorial types of) triples of pairs of letters which correspond to positions of spread out simplices. (For instance AB, AC, AD is missing because these would correspond to three simplices in a simplex of size 2.)

By duality and Remark 8.3, we will be done if, for each such triple, we can build a dual acyclic system σ^* of $d\Delta_2$ such that these are the pairs of labels adjacent to each of the three vertices of $d\Delta_2$. This is also done in Figure 22. The duals to those acyclic systems give rise to the desired simplex positions. \square

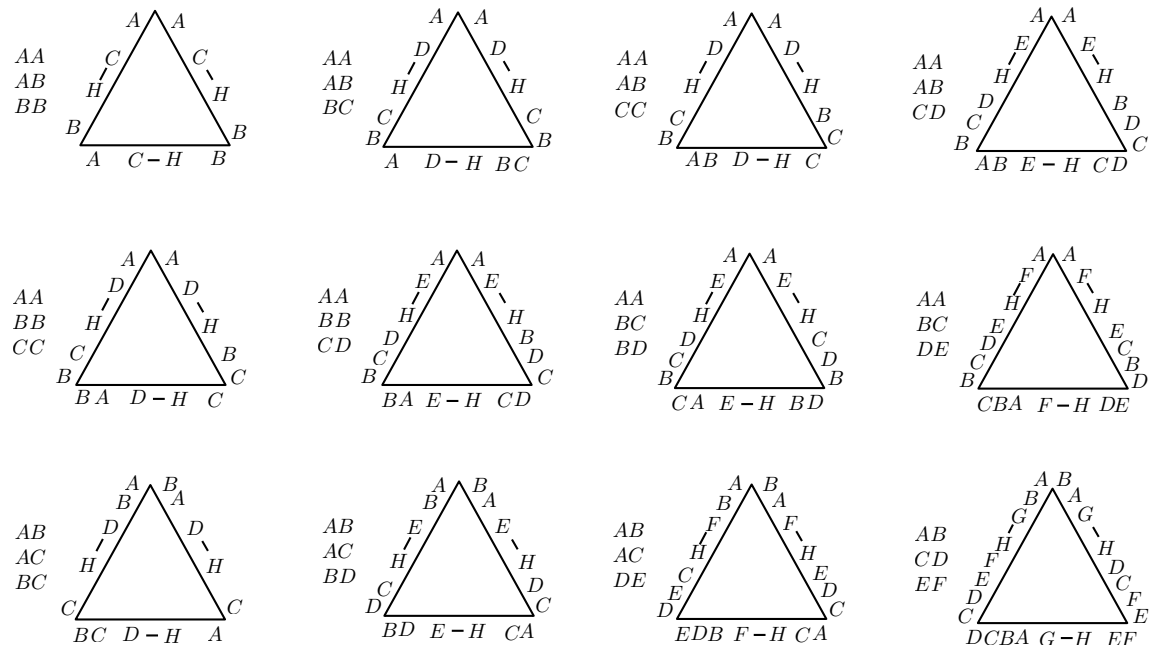


Figure 22: Combinatorial types of the positions of any three spread out simplices in $3\Delta_{d-1}$, and (non-unique) dual acyclic systems which generate such positions.

Theorem 8.11 (Spread Out Simplices Conjecture 8.1 for $n = 3$). *Three simplices in $3\Delta_{d-1}$ can be extended to a fine mixed subdivision if and only if they are spread out.*

Proof. This result is a consequence of the previous two theorems. \square

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References

- [1] F. Ardila, M. Beck, S. Hosten, J. Pfeifle, K. Seashore. *Root polytopes and growth series of root lattices*. SIAM Journal of Discrete Mathematics 25 (2011) 360-378.

- [2] F. Ardila and S. Billey. *Flag arrangements and triangulations of products of simplices*. Advances in Mathematics. 214 (2007), 495-524.
- [3] F. Ardila and M. Develin. *Tropical hyperplane arrangements and oriented matroids*. Math. Z. 262 (2009), 795-816.
- [4] E. Babson and L. Billera. *The geometry of products of minors*. Discrete Comput. Geom. 20 (1998), 231-249.
- [5] M. Bayer. *Equidecomposable and weakly neighborly polytopes*. Israel J. Math. 81 (1993), 301-320.
- [6] S. Billey and R. Vakil. Intersections of Schubert varieties and other permutation array schemes. IMA Volumes in Mathematics and its Applications Volume 146: Algorithms in Algebraic Geometry (2007) 21–54.
- [7] Jesús A. De Loera, Jörg Rambau and Francisco Santos. *Triangulations: Structures for Algorithms and Applications*. Algorithms and Computation in Mathematics, Vol. 25. Springer-Verlag, 2010.
- [8] M. Develin and B. Sturmfels. *Tropical convexity*. Documenta Math. 9 (2004), 1-27.
- [9] I. M. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, resultants and multidimensional determinants*, Birkhäuser, Boston, 1994.
- [10] M. Haiman. *A simple and relatively efficient triangulation of the n -cube*. Discrete Comput. Geom. 6 (1991) 287-289.
- [11] S. Herrmann, A. Jensen, M. Joswig and B. Sturmfels. *How to draw tropical planes*. Electron. J. Combin. 16 (2009). Special volume in honor of Anders Björner, Research Paper 6, 26 pp.
- [12] M. Luby, D. Randall, and A. Sinclair. Markov chain algorithms for planar lattice structures. *IEEE Symposium on Foundations of Computer Science* (1995) 150-159.
- [13] S. Oh and H. Yoo. *Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and Tropical Oriented Matroids*. Preprint arXiv:1009.4750.
- [14] D. Orden and F. Santos. *Asymptotically efficient triangulations of the d -cube*. Discrete Comput. Geom. 30 (2003), 509-528.
- [15] J. Oxley. *Matroid Theory*. Oxford University Press. New York, 1992.
- [16] F. Santos. *A point configuration whose space of triangulations is disconnected*. J. Amer. Math. Soc. 13 (2000), 611-637.

- [17] F. Santos. *Non-connected toric Hilbert schemes*. Mathematische Annalen 332 (2005), 645-665.
- [18] F. Santos. *The Cayley trick and triangulations of products of simplices*. Integer points in polyhedra - geometry, number theory, algebra, optimization. vol. 374 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2005. 151-177
- [19] B. Sturmfels. *Grobner Bases and Convex Polytopes*. American Mathematical Society, Univ. Lectures Series, No 8, Providence, 1996.
- [20] H. Yoo. Personal communication, 2010.